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Characters of Finite Semigroups

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Let S be a finite semigroup and let $J_1, \ldots, J_r$ be the regular $\mathscr{J}$-classes of S. Then the main theorem of this paper shows that

$$\text{ch}\, S \approx \text{ch}\, H_1 \times \cdots \times \text{ch}\, H_r$$

where, for example, ch S denotes the character ring of S and $H_1, \ldots, H_r$ are maximal subgroups of $J_1, \ldots, J_r$, respectively. As a consequence of this result, two representations of S are equivalent if and only if they are equivalent on the subgroups of S. Further, we show that each character of S can be uniquely expressed as an integral linear combination of what we term standard irreducible characters. As a consequence of this, the analog of an important theorem of Brauer [1] holds for finite semigroups; namely, every character of a finite semigroup can be expressed as an integral linear combination of characters induced from linear characters of its elementary subgroups.

1. Introduction

In this paper, we shall assume that all representations are over the complex field C and that all semigroups being considered are finite. If $\Gamma : S \to \text{hom}(A, A)$ is a representation of S, by linear transformations of a finite-dimensional vector space A over C, then the character χ_Γ of Γ is the function $S \to C$ defined by

$$\chi_\Gamma(x) = \text{Tr}\, \Gamma(x) \qquad \text{for each } x \in S.$$

where Tr $\Gamma(x)$ denotes the sum of the diagonal entries in any matrix expression of $\Gamma(x)$. Since $\text{Tr}\, AB = \text{Tr}\, BA$ for any square matrices A, B, χ_Γ is independent of the particular matrix form chosen for Γ.

The importance of characters stems from the fact that two representations Γ and Δ of S have (within equivalence) the same nonnull irreducible constituents if and only if $\chi_\Gamma = \chi_\Delta$ [10, p. 174–175]. In particular, fully reducible representations are equivalent if and only if their characters are equal. Thus, the representations of inverse semigroups are determined by their characters.

It follows from [9] and [5] that the irreducible representations of a finite semigroup S are determined by irreducible representations of its maximal subgroups. Thus, one would expect the characters of S to be intimately related to those of its subgroups. Theorem 3.4 shows that extent of this relationship for arbitrary finite semigroups; when the semigroup in question is 0-simple, there is a complete identification between characters of S and those of its maximal subgroups.

THEOREM 1.1 (Munn [7]). *Let $S = \mathscr{M}^\circ(G; m, n; P)$ be a finite 0-simple semigroup and let Γ be a proper representation of S which extends γ; then*

$$\chi_\Gamma(x; i, \lambda) = \chi_\gamma(p_{\lambda i}x) \qquad \text{for each } (x, i, \lambda) \in S.$$

The character of a representation Γ of S is easily seen to be the sum of the characters of its irreducible constituents. Hence, the structure of the irreducible representations of S is of prime importance in what follows. In the remainder of this section, we review those aspects of the representation theory of finite semigroups which we shall need in the sequel. The results are implicit in [5, 7, and 9] and are explicitly taken from [6].

Let S be a finite semigroup and let H be a maximal subgroup of S with identity e; let R be the $\mathscr{R}$-class containing H and J the corresponding $\mathscr{J}$-class. Let $\{q_\lambda : \lambda \in \Lambda\}$ be representatives of the $\mathscr{H}$-classes contained in R with $e = q_1$ the representative of H and, for each $x \in S$, let $M_J(x)$ be the $\Lambda \times \Lambda$ matrix over H° with λ, μ-th entry

$$M_J(x)_{\lambda,\mu} = \begin{Bmatrix} h \in H & \text{if} \quad q_\lambda x = hq_\mu \\ 0 & \text{if} \quad q_\lambda x \notin Hq_\mu \end{Bmatrix}.$$

Then, $M_J(x)$ is a row monomial matrix over H°, and the mapping $M_J : x \to M_J(x)$ is a homomorphism. Up to equivalence, M_J depends only on J; it is the Schützenberger representation of S corresponding to J [3, Theorem 3.16].

If $\{r_i : i \in I\}$ are representatives of the $\mathscr{H}$-classes contained in the $\mathscr{L}$-class of H, with $r_1 = e$ the representative of H, then each element of J can be uniquely expressed as

$$r_i g q_\lambda \qquad \text{where} \quad g \in H, \quad i \in I, \quad \lambda \in \Lambda.$$

The mapping $\psi : J^\circ \to M^\circ(H; I, \Lambda; P)$ defined by

$$x\psi = (g; i, \lambda) \qquad \text{if} \quad x = r_i g q_\lambda$$

is an isomorphism of J° onto the completely 0-simple semigroup

$\mathscr{M}^\circ(H; I, \Lambda; P)$, where $p_{\lambda i} = q_\lambda r_i \in H^\circ$ and J° denotes the semigroup obtained by adjoining a zero to J in the obvious way. The matrix P is called the *structure* matrix of J; it is uniquely determined by J up to equivalence.

In what follows, when we are dealing with a regular $\mathscr{J}$-class J of S, we shall assume that a maximal subgroup H and elements $\{q_\lambda : \lambda \in \Lambda\}$, $\{r_i : i \in I\}$ have been chosen and that M_J and P are defined in terms of these.

If γ is a proper representation of H then we obtain a proper representation γ_s of S by defining

$$\gamma_s(x) = \gamma(M_J(x)) \qquad \text{for each } x \in S,$$

where, for any matrix A over H°, $\gamma(A)$ denotes the matrix obtained by applying γ to each entry of A; γ_s is called the *standard extension* of γ to S, or a standard representation of S.

LEMMA 1.2. *With the notation above,* $\gamma_s(x) = 0$ *if and only if*

$$x \in U(J) = \{y \in S : S^1 y S^1 \cap J = \square\}.$$

Proof. This is straightforward.

We say that a representation Γ of S has apex J, a regular $\mathscr{J}$-class of S, if $\Gamma(x) \neq 0$ if and only if $x \notin U(J)$. Thus Lemma 1.2 states that γ_s has apex J. If the apex of Γ exists it is clearly unique.

THEOREM 1.3. *Let S be a finite semigroup.*

(A) *Every nonnull irreducible representation of S has an apex.*

(B) *Let J be a regular $\mathscr{J}$-class of S and let H_J be a maximal subgroup of J. For each proper representation γ of H_J let Q, Q^* be such that Q is epic, $Q^*\gamma(P)$ is monic and $QQ^*\gamma(P) = \gamma(P)$ (such always exist) and define*

$$B(\gamma)(x) = Q^*\gamma_s(x)Q \qquad \text{for each } x \in S.$$

Then $B(\gamma)$ is a proper representation of S with apex J. Further, if $\{\gamma_i : 1 \leqslant i \leqslant n\}$ is a full set of inequivalent irreducible representations of H, then $B(\gamma_i) : \{1 \leqslant i \leqslant n\}$ is a full set of inequivalent irreducible representations of S with apex J.

(C) *Each irreducible representation of an ideal of S has a unique extension to an (irreducible) representation of S.*

(D) *Each representation of S is fully reducible if and only if S is regular and the structure matrix of J is invertible over $C(H_J)$ for each $\mathscr{J}$-class J. In this case, $B(\gamma) = \gamma_s$ for each irreducible representation γ of a maximal subgroup of S. Thus each representation of S is a direct sum of standard representations.*

Let H be a maximal subgroup of S and let γ be a proper representation of H with character χ. Then, we shall denote by χ_s the character of γ_s and by $B(\chi)$ the character of $B(\gamma)$; thus, the irreducible characters of S are the characters $B(\chi)$, where χ ranges over the irreducible characters of the maximal subgroups of S.

2. Conjugacy in Finite Semigroups

If G is a finite group, then two elements $a, b \in G$ are conjugate if and only if $\chi(a) = \chi(b)$ for each (irreducible) character χ of G. The situation for finite semigroups is similar but rather more complicated; to describe it we need the following lemma.

Lemma 2.1. *Let S be a finite semigroup; let $a \in S$ and let $\bar{a} = ae$, where e is the unique idempotent in the subsemigroup $\langle a \rangle$ generated by a. Then $\chi(a) = \chi(\bar{a})$ for each character χ of S.*

Proof. Let Γ be a representation of S of degree n. Then, relative to a suitable basis,

$$\Gamma(e) = \begin{bmatrix} I_t & 0 \\ 0 & 0 \end{bmatrix}$$

for some $t \leqslant n$. Suppose $\Gamma(a)$ has the block form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

relative to this basis. Then, since a and e commute, $B = 0$, $C = 0$. Further, since $a^m = e$ for some m, D is nilpotent and so has trace zero. Thus

$$\operatorname{Tr} \Gamma(a) = \operatorname{Tr} A = \operatorname{Tr} \Gamma(a)\, \Gamma(e) = \operatorname{Tr} \Gamma(\bar{a}).$$

Theorem 2.2. *Let a, b be elements of a finite semigroup S. Then $\chi(a) = \chi(b)$ for each character χ of S if and only if*

$$\bar{b} = x'\bar{a}x, \qquad \bar{a} = x\bar{b}x'$$

for some regular element x with inverse x'.

Proof. Suppose $\bar{b} = x'\bar{a}x$, $\bar{a} = x\bar{b}x'$, where x, x' are inverse to one another. Then $\bar{a} = xx'\bar{a}xx'$ so that, since xx' is idempotent, $\bar{a} = xx'\bar{a}$. Hence, by the general properties of the trace

$$\chi(b) = \chi(\bar{b}) = \chi(x'\bar{a}x) = \chi(xx'\bar{a}) = \chi(\bar{a}) = \chi(a)$$

for any character χ.

Conversely, if $\chi(a) = \chi(b)$ for all characters χ of S, it follows from Lemma 2.1 that $\chi(\bar{a}) = \chi(\bar{b})$ for each χ. Suppose that $(\bar{a}, \bar{b}) \notin \mathscr{J}$; then, without loss of generality, we may assume that $\bar{b} \in U(J)$ where J is the (regular) $\mathscr{J}$-class containing $\bar{a}$. Let H be the group $\mathscr{H}$-class containing $\bar{a}$ and let u be the trivial representation of H. Then $\Gamma = u_s$, the standard extension of u to S, is a representation of S such that $\Gamma(b) = 0$ and $\Gamma(\bar{a})$ is a nonzero idempotent. Hence $\chi_\Gamma(a) \neq \chi_\Gamma(b)$ which contradicts the assumption on a, b. Thus $(\bar{a}, \bar{b}) \in \mathscr{J}$.

Since the $\mathscr{J}$-class J containing $\bar{a}$, $\bar{b}$ is regular, it follows from Theorem 1.3 (B) that $\chi(\bar{a}) = \chi(\bar{b})$ for each character χ of J°. Suppose that $J^\circ = \mathscr{M}^\circ(H; I, \Lambda; P)$ and that $\bar{a} = (u; i, \lambda)$, $\bar{b} = (v; j, \mu)$. Then, it follows from Theorem 1.1 that $p_{\lambda i}u$ is conjugate to $p_{\mu j}v$; say $p_{\mu j}v = g^{-1}p_{\lambda i}ug$ for some $g \in H$. (Note $p_{\lambda i} \neq 0$, $p_{\mu j} \neq 0$ since $\bar{a}$, $\bar{b}$ belong to a maximal subgroup of J.) Let

$$x = (p_{\lambda i}^{-1}g; i, \mu), \qquad x' = (p_{\mu j}^{-1}g^{-1}; j, \lambda).$$

Then x, x' are mutually inverse and

$$\bar{b} = x'\bar{a}x, \qquad \bar{a} = x\bar{b}x'.$$

COROLLARY 2.3. *Let a, b be elements of a finite inverse semigroup S. Then $\chi(a) = \chi(b)$ if and only if $\bar{b} = x^{-1}\bar{a}x$, $\bar{a} = x\bar{b}x^{-1}$ for some $x \in S$.*

We shall say that two elements a, b, in a semigroup S, are *conjugate* if and only if $\bar{b} = x'\bar{a}x$, $\bar{a} = x\bar{b}x'$ for some regular element x with inverse x'.

Given mutually inverse elements x, x' in a semigroup S one can define a partial inner automorphism $\phi_{(x,x')} : xx'Sxx' \to x'xSx'x$ by

$$a\phi_{(x,x')} = x'ax \qquad \text{for each } a \in xx'Sxx'.$$

Then it is easy to see that a is conjugate to b if and only if $\bar{b}$ is the image of $\bar{a}$ under some partial inner automorphism; this corresponds exactly with the situation for groups.

For the special case of the full transformation semigroup $\mathscr{T}_X$ on a finite set X, the analogy with the group case is particularly strong. Let $\alpha \in \mathscr{T}_X$; then

$$X \supseteq X\alpha \supseteq X\alpha^2 \supseteq$$

and so $X\alpha^n = X\alpha^{n+1}$ for some n so that $\alpha \mid X\alpha^n$ is a permutation of $X\alpha^n$. We define the cycle structure of α to be the cycle structure of this permutation.

THEOREM 2.4. *Let X be a finite set and let $\alpha, \beta \in \mathscr{T}_X$. Then α and β are conjugate if and only if they have the same cycle structure.*

Proof. If $X\alpha^n = X\alpha^{n+1}$ and θ denotes the idempotent in $\langle a \rangle$, then θ is the identity when restricted to $X\alpha^n$. Thus α and $\bar{\alpha} = \alpha\theta$ have the same cycle structure. Hence we may, without loss of generality, assume that α, β are permutations on $X\alpha$, $X\beta$ respectively; that is, $\alpha = \bar{\alpha}$, $\beta = \bar{\beta}$.

If α, β have the same cycle structure, then exactly as in the case of the symmetric group there is a one-to-one onto mapping $\phi : X\alpha \to X\beta$ such that

$$\begin{tikzcd} X\alpha \arrow[r, "\alpha"] \arrow[d, "\phi"'] & X\alpha \arrow[d, "\phi"] \\ X\beta \arrow[r, "\beta"] & X\beta \end{tikzcd}$$

commutes. Define $\psi : X \to X$, $\eta : X \to X$ by

$$x\psi = x\beta\phi^{-1}\alpha^{-1} \qquad x\eta = x\alpha\phi\beta^{-1}$$

for each $x \in X$, where, for example, $x\alpha^{-1}$ denotes the image of $x \in X\alpha$ under the inverse of $\alpha \mid X\alpha$. Then,

$$\beta = \psi\alpha\eta, \qquad \alpha = \eta\beta\psi$$

where η, ψ are inverses of one another.

Conversely, if $\beta = \psi\alpha\eta$, $\alpha = \eta\beta\psi$ where η, ψ are mutually inverse, then $\alpha = \eta\psi\alpha\eta\psi$ so that, since $\eta\psi$ is idempotent, it is the identity on $X\alpha$. Hence

$$X\beta \subseteq X\alpha\eta \subseteq X\eta\psi\alpha\eta \subseteq X\beta$$

so that η maps $X\alpha$ onto $X\beta$ and similarly ψ maps $X\beta$ onto $X\alpha$. Hence, $|X\alpha| = |X\beta|$, and therefore η maps $X\alpha$ one-to-one onto $X\beta$. For $x \in X\alpha$,

$$x\alpha\eta = x\eta\psi\alpha\eta = x\eta\beta.$$

Hence, exactly as for the symmetric group, α, β have the same cycle structure.

An argument like that used in the proof of Theorem 2.4 shows that two elements, of the symmetric inverse semigroup on a finite set, are conjugate if and only if they have the same cycle structure. This result was obtained by Munn [8].

A finite group has precisely as many inequivalent irreducible representations as it has conjugacy classes. It follows immediately from Theorems 1.3 and 2.2 that the same is true for finite semigroups; specifically we have the following result.

PROPOSITION 2.5. *Let $J_1, \ldots, J_r$ be the regular $\mathcal{J}$-classes of a finite semigroup S and let $H_1, \ldots, H_r$ be maximal subgroups of $J_1, \ldots, J_r$ respectively.*

Suppose that for each $i = 1,\ldots, r$, $x_1^i,\ldots, x_{r_i}^i$ *are representatives of the conjugacy classes of* H_i. *Then* $x_{j_i}^i : 1 \leqslant j_i \leqslant n_i$, $1 \leqslant i \leqslant r$ *are representatives of the conjugacy classes of* S.

3. The Character Ring

Let S be a finite semigroup; then a function $S \to C$ is called a *class function* if it is constant on each conjugacy class of S. Thus, by Theorem 2.2, the characters of S are class functions. Class functions clearly form an algebra over C under pointwise addition and multiplication; we denote this algebra by cf S.

The irreducible characters of S are linearly independent functions $S \to C$. (This follows easily by an argument like that used in [10, theorem on p. 174].) Hence, since, by Proposition 2.5, there are precisely as many irreducible representations as conjugacy classes, the irreducible characters form a basis for cf S.

The following proposition gives a simple characterisation of the members of cf S.

Proposition 3.1. *Let* S *be a finite semigroup. Then a function* $f : S \to C$ *is a class function if and only if*

(i) $f(a) = f(\bar{a})$ *for each* $a \in S$;
(ii) $f(ab) = f(ba)$ *for each* $a, b \in S$.

Proof. If (i) and (ii) hold then an argument identical to that used in the first part of the proof of Theorem 2.2 shows that f is a class function.

Conversely, since a class function is a linear combination of characters and (i) and (ii) hold for each character of S, it follows that (i) and (ii) hold for each $f \in$ cf S.

If Γ and Δ are representations of a finite semigroup S, then the direct sum $\Gamma \oplus \Delta$ of Γ, Δ has character $\chi_\Gamma + \chi_\Delta$. Thus the set of characters (including the zero character) is just the set of nonnegative integral linear combinations of irreducible characters of S. The set of differences between characters is thus the subgroup of cf S spanned by the irreducible characters; we denote this subgroup by ch S. The characters of S admit a natural multiplication, for if Γ and Δ are representations of S, then their tensor product $\Gamma \otimes \Delta$ is a representation of S with character $\chi_\Gamma\chi_\Delta$. This is just the pointwise multiplication of functions; so ch S is a subring of cf S; we call ch S the ring of generalized characters of S or the *character ring* of S.

The main theorem of this section characterises the character ring of S in terms of the character rings of its maximal subgroups. To prove this result, we require the following lemma.

LEMMA 3.2. *Let J be a regular $\mathscr{J}$-class of a finite semigroup S and let H be a maximal subgroup of J. Let χ be a linear combination of irreducible characters of S, each with apex J. If $\chi \mid H \in \text{ch } H$ then $\chi \in \text{ch } S$.*

Proof. Suppose that $\chi = \alpha_1\chi_1 + \cdots + \alpha_n\chi_n$ where $\chi_1, \ldots, \chi_n$ are the irreducible characters of S with apex J. Then

$$\chi \mid H = \alpha_1\chi_1 \mid H + \cdots + \alpha_n\chi_n \mid H,$$

where, by Theorem 1.3 (B), $\chi_1 \mid H, \ldots, \chi_n \mid H$ are the nonzero irreducible characters of H. Since these functions are a basis for cf H and $\chi \mid H \in \text{ch } H$, it follows that $\alpha_1, \ldots, \alpha_n$ are integers. Thus $\chi \in \text{ch } S$.

COROLLARY 3.3. *Let $\chi = \alpha_1\chi_1 + \cdots + \alpha_r\chi_r$ be a linear combination of irreducible characters of S and suppose that $\chi \mid U \in \text{ch } U$ for some ideal U of S which contains the apex of each χ_i. Then $\chi \in \text{ch } S$.*

Proof. We use induction on r. If $r = 1$ or each χ_i has the same apex, the result is immediate from Lemma 3.2. If this is not the case suppose that $\chi_1, \ldots, \chi_k$ have apex J and J is minimal among the apexes of $\chi_1, \ldots, \chi_r$. Then, $\chi_j(J) = 0$ for $k + 1 \leqslant j \leqslant r$ so that Lemma 3.2 may be applied to show $\alpha_1, \ldots, \alpha_k$ are integers. But this gives $\alpha_{k+1}\chi_{k+1} + \cdots + \alpha_r\chi_r \mid U \in \text{ch } U$ and so, by induction on r, $\alpha_{k+1}, \ldots, \alpha_r$ are integers.

THEOREM 3.4. *Let S be a finite semigroup, let $J_1, \ldots, J_r$ be the regular $\mathscr{J}$-classes of S, and let $H_1, \ldots, H_r$ be maximal subgroups of $J_1, \ldots, J_r$, respectively. Then,* $\text{ch } S \approx \text{ch } H_1 \times \cdots \times \text{ch } H_r$.

Proof. Let $\chi \in \text{cf } S$ and define $\chi\theta \in \text{cf } H_1 \times \cdots \times \text{cf } H_r$ by $\chi\theta = (\chi_1, \ldots, \chi_r)$, where $\chi_i = \chi \mid H_i$. Then, it is easy to see that, for $\chi, \psi \in \text{cf } S$, $\alpha \in C$,

$$\chi_i + \psi_i = (\chi + \psi)_i, \qquad (\chi\psi)_i = \chi_i\psi_i, \qquad (\alpha\chi)_i = \alpha\chi_i$$

so that θ is a homomorphism of cf S into cf $H_1 \times \cdots \times \text{cf } H_r$.

Suppose that $\chi \in \ker \theta$. Without loss of generality, we may suppose $\chi = \chi^{(1)} + \cdots + \chi^{(r)}$ where $\chi^{(i)}$ is a linear combination of irreducible characters each with apex J_i. Further we can suppose that $i \leqslant j$ implies $J_j \nleqslant J_i$. Then, since $J_i \nleqslant J_1$, $2 \leqslant i \leqslant r$, $J_1 \subseteq U(J_i)$ for $2 \leqslant i \leqslant r$. Hence,

$$\chi(x) = \chi^{(1)}(x) \qquad \text{for each } x \in J_1 .$$

In particular, $\chi_1 = \chi_1^{(1)}$ so that $\chi_1^{(1)} = 0$. This clearly implies $\chi^{(1)} = 0$. Since $S^1J_1S^1 \subseteq U(J_i)$, $2 \leqslant i \leqslant r$, we can thus regard χ as a member of cf T where $T = S/S^1J_1S^1$. Now $J_2, \ldots, J_r$ are the regular $\mathscr{J}$-classes of T so that $\chi = \chi^{(2)} + \cdots + \chi^{(r)}$ belongs to the kernel of the appropriate homomorphism

$\mathrm{cf}\, T \to \mathrm{cf}\, H_2 \times \cdots \times \mathrm{cf}\, H_r$. Thus, by induction on r (or repetition), $\chi^{(2)} = 0 = \cdots = \chi^{(r)}$. Hence, $\chi = 0$ and θ is one to one. Since the dimension of $\mathrm{cf}\, S =$ dimension of $\mathrm{cf}\, H_1 \times \cdots \times \mathrm{cf}\, H_r$, by Proposition 2.5, it follows that θ is an isomorphism.

The isomorphism θ clearly maps $\mathrm{ch}\, S$ into $\mathrm{ch}\, H_1 \times \cdots \times \mathrm{ch}\, H_r$ so we need only show that θ, regarded as a map $\mathrm{ch}\, S \to \mathrm{ch}\, H_1 \times \cdots \times \mathrm{ch}\, H_r$ is onto. Let $(\chi_1, ..., \chi_r) \in \mathrm{ch}\, H_1 \times \cdots \times \mathrm{ch}\, H_r$. Then, since θ is an isomorphism of $\mathrm{cf}\, S$ onto $\mathrm{cf}\, H_1 \times \cdots \times \mathrm{cf}\, H_r$, there exists $\chi \in \mathrm{cf}\, S$ such that $\chi\theta = (\chi_1, ..., \chi_r)$. Without loss of generality, we can assume, as above, that $\chi = \psi_1 + \cdots + \psi_r$, where ψ_i is a linear combination of irreducible characters each with apex J_i, $1 \leqslant i \leqslant r$. Since J_r is a maximal regular $\mathcal{J}$-class, $U(J_r)$ is an ideal of S with regular $\mathcal{J}$-classes $J_1 \cdots J_{r-1}$. Further, $\chi \mid U(J_r)$ corresponds to the member $(\chi_1, ..., \chi_{r-1}) \in \mathrm{ch}\, H_1 \times \cdots \times \mathrm{ch}\, H_{r-1}$. Hence, by induction on r, $\chi \mid U(J_r) \in \mathrm{ch}\, U(J)$; note that, if $r = 1$, the result is immediate from Lemma 3.2.

For each $x \in U(J_r)$

$$\chi(x) = \psi_1(x) + \cdots + \psi_{r-1}(x)$$

so, that since $\psi_i \mid U(J_r)$ is a linear combination of irreducible characters of $U(J_r)$, $\psi_i \mid U(J_r)$ belongs to $\mathrm{ch}\, U(J_r)$, $1 \leqslant i \leqslant r-1$. Since apex $\psi_i = J_i \subseteq U(J_r)$, it follows from Corollary 2.2 that $\bar{\chi} = \psi_1 + \cdots + \psi_{r-1} \in \mathrm{ch}\, S$. Now, for each $h \in H_r$,

$$\chi(h) = \bar{\chi}(h) + \psi_r(h)$$

so that $\psi_r \mid H_r \in \mathrm{ch}\, H_r$. Hence, by Lemma 2.1, $\psi_r \in \mathrm{ch}\, S$; thus $\chi \in \mathrm{ch}\, S$.

COROLLARY 3.5. *Let S be a finite semigroup with regular $\mathcal{J}$-classes $J_1, ..., J_r$ and let $H_1, ..., H_r$ be maximal subgroups of $J_1, ..., J_r$ respectively. Then two representations Γ and Δ of S over C have the same irreducible constituents if and only if $\Gamma \mid H_i$ is equivalent to $\Delta \mid H_i$, $1 \leqslant i \leqslant r$.*

The proof of Theorem 3.4 shows that a class function χ on a finite semigroup S is a generalised character of S if and only if its restriction to each maximal subgroup H of S is a generalised character of H. This result is similar to a result of Brauer [1] which characterizes the characters of a group in terms of so called elementary subgroups; a group G is called elementary if it is the direct product of a cyclic group and a p-group for some prime p. Indeed, using Brauer's theorem, we have

COROLLARY 3.6. *Let S be a finite semigroup and let χ be a class function on S. Then $\chi \in \mathrm{ch}\, S$ if and only if $\chi \mid K \in \mathrm{ch}\, K$ for each elementary subgroup K of S.*

In Section 5, we shall prove an analog of another result of Brauer [1], namely, that every character of a finite semigroup S is an integral linear combination of characters induced from linear characters of the elementary subgroups of S.

The mapping $\theta : \text{ch } S \to \text{ch } H_1 \times \cdots \times \text{ch } H_r$ clearly maps characters of S onto n-tuples of characters of $H_1, \ldots, H_r$. In general, θ does not map the characters of S onto the set of n-tuples of characters of $H_1, \ldots, H_r$. In fact:

THEOREM 3.7. *Let S be a finite semigroup and let $H_1, \ldots, H_r$ be maximal subgroups of the distinct regular $\mathscr{J}$-classes of S. Then θ maps the set of characters of S onto the set of characters of $H_1, \ldots, H_r$ if and only if S has no comparable regular $\mathscr{J}$-classes.*

Proof. Suppose that J_i, J_j are regular $\mathscr{J}$-classes of S with $J_i < J_j$ and let χ be a character of S such that $\chi(e_i) \neq 0$, where e_i is the identity of $H_i \subseteq J_i$; then clearly, $\chi(e_j) \neq 0$ where e_j is the identity of H_j. By the hypothesis on S, there is a character χ such that, in the notation of Theorem 3.3,

$$\chi_j(h) = \begin{cases} 1 & \text{if } \ i = j \\ 0 & \text{if } \ i \neq j. \end{cases}$$

Hence S has no comparable regular $\mathscr{J}$-classes.

Conversely, let $(\chi_1, \ldots, \chi_r) \in K(H_1) \times \cdots \times K(H_r)$ where $K(H_i)$ denotes the set of characters of H_i. Let $B(\chi_1), \ldots, B(\chi_r)$ be the characters of the basic extensions to S of representations of $H_1, \ldots, H_r$ with characters $\chi_1, \ldots, \chi_r$ respectively. Then, since $J_i \subseteq U(J_j)$ if $i \neq j$,

$$B(\chi_i) \mid H_j = \begin{cases} \chi_i & \text{if } \ j = i \\ 0 & \text{if } \ i \neq j. \end{cases}$$

Hence $\chi = B(\chi_1) + \cdots + B(\chi_r)$ is a character of S with $\chi\theta = (\chi_1, \ldots, \chi_r)$.

COROLLARY 3.8. *Let S be a finite regular semigroup. Then θ is onto $K(H_1) \times \cdots \times K(H_r)$ if and only if S is an* 0*-direct union of* 0*-simple semigroups.*

4. DECOMPOSITION OF CHARACTERS

If G is a finite group then one can define an inner product $\langle\ ,\ \rangle$ on cf G as follows

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_G \chi(g) \overline{\psi(g)}.$$

With respect to this inner product, the irreducible characters of G form an orthonormal basis for cf G. Hence when this inner product is restricted to ch G, one has a method for finding the irreducible constituents of a representation Γ of G. For

$$\chi_\Gamma = \langle \chi_\Gamma , \chi_1 \rangle \chi_1 + \cdots + \langle \chi_\Gamma , \chi_r \rangle \chi_r ,$$

where $\chi_1 ,\ldots, \chi_r$ are the irreducible characters of G.

If $S = M^\circ(G; m, n; P)$ is a finite 0-simple semigroup, then one can extend the inner product on cf G to one on cf S with respect to which the irreducible representations form an orthonormal basis. For, let Γ, Δ be respectations of S which extend γ, δ respectively; then,

$$\begin{aligned}\sum_S \chi_\Gamma(s) \overline{\chi_\Delta(s)} &= \sum_S \chi_\gamma(p_{\lambda i}x) \overline{x_\delta(p_{\lambda i}x)} \\ &= \sum_{\substack{(\lambda, i) \\ p_{\lambda i} \neq 0}} \sum_G \chi_\gamma(g) \overline{x_\delta(g)} \\ &= | E | \, | G | \langle \chi_\gamma , \chi_\delta \rangle,\end{aligned}$$

where E denotes the set of (nonzero) idempotents of S. Hence, if we define

$$\langle \chi_\Gamma , \chi_\Delta \rangle = \frac{1}{| E | \, | G |} \sum_S \chi_\Gamma(x) \overline{\chi_\Delta(x)},$$

we get an inner product on cf S with respect to which the irreducible characters are orthonormal. These results are due to Munn [7].

In general, there does not seem to be a similar natural inner product on cf S when S is an arbitrary finite semigroup; so we adopt another approach to the problem of decomposing a character of S into its irreducible constituents.

Suppose that $H_1 ,\ldots, H_r$ are maximal subgroups of the distinct regular $\mathcal{J}$-classes of S and suppose that, for $1 \leqslant i \leqslant r$, χ_{i,λ_i}, $1 \leqslant \lambda_i \leqslant n_i$ are the irreducible characters of H_i. Then $\chi'_{i,\lambda_i} = B(\chi_{i,\lambda_i})$ are the irreducible characters of S. By Theorem 2.3, the mapping $\theta : \text{cf } S \to \text{cf } H_1 \times \cdots \times \text{cf } H_r$ defined by

$$\chi\theta = (\chi^1,\ldots, \chi^r),$$

where $\chi^i = \chi \mid H_i$, is an isomorphism which maps ch S onto

$$\text{ch } H_1 \times \cdots \times \text{ch } H_r .$$

The matrix $\mathscr{B}^{-1}$ of θ relative to the bases χ'_{i,λ_i} and $(0 \cdots \chi_{i,\lambda_i} \cdots 0)$ is given by

$$\mathscr{B}^{-1}_{(i,\lambda_i),(j,\mu_j)} = \langle \chi'_{i,\lambda_i} \mid H_j , \chi_{j,\mu_j} \rangle_j ,$$

where $\langle \ , \ \rangle_j$ denotes the inner product on H_j, $1 \leqslant j \leqslant r$; thus

$$\mathscr{B}^{-1}_{(i,\lambda_i),(j,\mu_j)} = \frac{1}{|H_j|} \sum \chi'_{i,\lambda_i}(g) \overline{\chi_{j,\mu}(g)}.$$

We shall call $\mathscr{B}$ (the inverse of $\mathscr{B}^{-1}$) the *character decomposition matrix* of S.

THEOREM 4.1. *Let S be a finite semigroup and let $H_1 ,..., H_r$ be maximal subgroups of the distinct regular $\mathscr{J}$-classes of S. Suppose that χ_{i,λ_i}, $1 \leqslant \lambda_i \leqslant n_i$, are the irreducible characters of H_i, $1 \leqslant i \leqslant r$, and let $\mathscr{B}$ be the character decomposition matrix of S. If χ is a character of S, then*

$$\chi = \sum_1^r \sum_1^{n_i} a_{i,\lambda_i} \chi'_{i,\lambda_i}$$

where

$$(a_{11} ,..., a_{r,n_r}) = (c_{11} ,..., c_{r,n_r})\mathscr{B}$$

with

$$c_{i,\lambda_i} = \langle \chi^i, \chi_{i,\lambda_i} \rangle_i = \frac{1}{|H_i|} \sum_{H_i} \chi(g) \overline{\chi_{i,\lambda_i}(g)}.$$

Proof. By the definition of θ, $(c_{11} ,..., c_{r,n_r})$ are the coordinates of $\chi\theta$ relative to the basis $(0 \cdots \chi_{i,\lambda_i} \cdots 0)$ for $\mathrm{ch}\, H_1 \times \cdots \times \mathrm{ch}\, H_r$. Hence $(c_{11} ,..., c_{r,n_r})\mathscr{B}$ gives the coordinates of χ relative to the basis χ'_{i,λ_i} for ch S.

If we suppose that the $\mathscr{J}$-classes $J_1 ,..., J_r$ in Theorem 4.1 are ordered so that $i < j$ implies $J_j \nleqslant J_i$, then it is easy to see that $\mathscr{B}^{-1}$ is an upper triangular matrix with diagonal entries 1. Thus $\mathscr{B}$ is also upper triangular and the entries of B are integral polynomials in the entries of $\mathscr{B}^{-1}$. Hence, since the latter are integers, the entries of $\mathscr{B}$ are integers.

5. STANDARD AND INDUCED CHARACTERS

The process described in Theorem 4.1 for decomposing characters of finite semigroups is just as explicit as the procedure for finite groups in terms of the inner product on characters. Further, bearing in mind the fact that one has to take into account several groups rather than just one, the procedure is not really much more complicated. However, it does have the drawback that, in order to construct the irreducible characters of a finite semigroup, one needs to know the irreducible representations. In the case of finite groups, there are methods for finding characters without actually constructing representations. It is an open question whether one can find a method for constructing the irreducible characters of semigroups without having to find the representations first.

Although the irreducible characters of an arbitrary finite semigroup are not immediately constructable from the characters of its maximal subgroups, one can give formulae for the standard characters directly in terms of group characters. Thus when the semigroup algebra is semisimple the character decomposition matrix can be explicitly described in terms of group characters.

Let J be a regular $\mathscr{J}$-class of a finite semigroup S and let H be a maximal subgroup of J. Then, for any proper representation γ of H, the standard extension γ_s of γ to S is defined by

$$\gamma_s(x) = \gamma(M_J(x)) \qquad \text{for each } x \in S.$$

Hence, the character χ_s of γ_s is given by

$$\chi_s(x) = \sum \{x(q_\lambda x q_\lambda{}^*) : q_\lambda x \in Hq_\lambda\}, \tag{5.1}$$

where q_λ, $q_\lambda{}^*$ are as in Section 1. This formula appears to depend on the choice of q_λ as well as one H and χ. However, since γ_s depends only on H and γ, the value of the character is independent of the particular choice of q_λ.

Let R be the $\mathscr{R}$-class of S containing H and for each $r \in R$, $x \in S$ define $rxr^{-1} \in H^\circ$ by

$$rxr^{-1} = \begin{cases} h \in H & \text{if } \ rx = hr \\ 0 & \text{if } \ rx \notin Hr; \end{cases} \tag{5.2}$$

note that, if J° is inverse, this coincides with the normal meaning of $rxr^{-1} \in S/U(J)$ so there is no reason for ambiguity.

THEOREM 5.1. *Let S be a finite semigroup and let χ be a character of a maximal subgroup H of S. Then*

$$\chi_s(x) = \frac{1}{|H|} \sum_R \chi(rxr^{-1}) \qquad \textit{for each} \quad x \in S, \tag{5.3}$$

where R is the $\mathscr{R}$-class of S containing H and χ is extended to H° by defining $\chi(0) = 0$.

Proof. Let $r \in R$; then $r = gq_\lambda$ for a unique $g \in H$, $\lambda \in \Lambda$. Hence, since $q_\lambda x = hq_\lambda$, $h \in H$ if and only if $rx = ghg^{-1}r$.

$$\chi(q_\lambda x q_\lambda{}^*) = \chi(h) = \chi(ghg^{-1}) = \chi(rxr^{-1})$$

when $q_\lambda x \in Hq_\lambda$ and so, since r varies over R as g varies over H and λ over Λ,

$$\chi_s(x) = \frac{1}{|H|} \sum_R \chi(rgr^{-1}).$$

It follows that when $C(S)$ is semisimple one can explicitly write down the irreducible characters of S in terms of the characters of its maximal subgroups. In particular, if S is inverse, the irreducible characters are given by Theorem 5.1 and in this case we may even take rxr^{-1} to have its usual meaning in S provided that χ is extended to S by defining $\chi(y) = 0$ for each $y \in S \backslash H$. Munn [9] essentially uses the fact that the irreducible characters of a finite inverse semigroup are standard characters to calculate all irreducible characters of the symmetric inverse semigroup on a finite set.

Ever when $C(S)$ is not semisimple standard characters appear in many natural situations. For example we have

THEOREM 5.2. *Let S be a finite regular semigroup and let $H_1, \ldots, H_r$ be maximal subgroups of the distinct $\mathscr{J}$-classes of S. Then the regular character ρ of S is given by*

$$\rho = n_1\rho_s^1 + \cdots + n_r\rho_s^r,$$

where n_i denotes the number $\mathscr{R}$-classes in the $\mathscr{J}$-class J_i containing H_i and ρ^i denotes the regular character of H_i.

Proof. By (5.3)

$$\rho_s^i(x) = \frac{1}{|H_i|} \sum_{R_i} \rho(rxr^{-1}).$$

Hence, since $\rho^i(h) = |H_i|, 0$ according as h is or is not the identity of H_i,

$$\begin{aligned}\rho_s^i(x) &= \frac{1}{|H_i|} \sum \{|H_i| : rx = r\} = |r \in R_i : rx = r| \\ &= \frac{1}{n_i} |s \in J_i : sx = s|.\end{aligned}$$

Thus, since S is regular

$$\begin{aligned}n_1\rho_s^1 + \cdots + n_r\rho_s^r(x) &= \sum_1^r |s \in J_i : sx = s| \\ &= |s \in S : sx = s| = \rho(x).\end{aligned}$$

COROLLARY 5.3. *Let S be a finite semigroup whose algebra over C is semisimple and let $\chi_1, \ldots, \chi_t$ be a full set of equivalent irreducible characters of S. Then*

$$\rho = m_1\chi_1 + \cdots + m_t\chi_t,$$

where m_i is the degree of χ_i, $1 \leqslant i \leqslant t$.

Proof. By Theorem 1.3 (D), S is regular; so we may apply Theorem 5.2. Let J be a $\mathscr{J}$-class of S with n $\mathscr{R}$-classes and let H be a maximal subgroup of J. Then if $\psi^1, \ldots, \psi^t$ denote the irreducible characters of H, the regular character of H is $\rho^H = k_1\psi^1 + \cdots + k_t\psi^t$ where $k_i =$ degree ψ^i. Then $\rho_s{}^H = k_1\psi_s{}^1 + \cdots + k_t\psi_s{}^t$ and so, since $\psi_s{}^i$ has degree nk_i,

$$n\rho_s{}^H = m_1\psi_s{}^1 + \cdots + m_t\psi_s{}^t,$$

where $m_i =$ degree $\psi_s{}^i$. Since $C(S)$ is semisimple, $\psi_s{}^1, \ldots, \psi_s{}^t$ are the irreducible characters of S with apex J and the result follows by considering each $\mathscr{J}$-class in turn.

The analog of Corollary 5.3 is known to hold in any finite-dimensional semisimple algebra over C [10].

It follows from Theorem 5.2, that the regular representation of a finite semigroup S with $C(S)$ semisimple is the direct sum of what we may call the regular standard representations of S. However, in general, the regular representation of a finite semigroup need not even be decomposable. For example, this is the case for $S = \{a, b, 1 : ab = a = a^2, ba = b = b^2\}$.

If $C(S)$ is semisimple then every character is a sum of standard characters. Again this is not the case in general (although it is the case for 0-simple semigroups). For example, if $S = \{a, b, 1 : ab = b = b^2, ba = a = a^2\}$, then the trivial representation of S has character χ given by $\chi(x) = 1$ for each $x \in S$. One can easily verify that

$$\chi = \psi_s - \phi_s,$$

where ψ is the unit representation of $\{a\}$ and ϕ that of $\{1\}$. Since ϕ_s, ψ_s are the only standard irreducible characters of S, χ cannot be a sum of standard characters. This example is typical as Theorem 5.5 shows that every character of S is an integral linear combination of standard characters.

LEMMA 5.4. *Let S be a finite semigroup, let $J_1, \ldots, J_r$ be the regular $\mathscr{J}$-classes of S, and suppose that $i \leqslant j$ implies $J_j \nleqslant J_i$. Then the standard irreducible characters (i.e., the standard extensions to S of irreducible characters of maximal subgroups of S) form a basis for* cf S.

Proof. Since there are as many standard irreducible characters as there are irreducible characters, we need only show that the standard irreducible characters are linearly independent. In fact, since the distinct standard irreducible characters with apex J induce the distinct irreducible characters on a maximal subgroup H of J, we need only show that, if $\chi_1 + \cdots + \chi_r = 0$, where χ_i is a linear combination of standard irreducible characters with apex J_i, then $\chi_i = 0$, $1 \leqslant i \leqslant r$.

Since J_1 is a minimal regular $\mathscr{J}$-class of S, $\chi_1 + \cdots + \chi_r = 0$ implies $\chi_1 \mid J_1 = 0$ and so $\chi_1 = 0$. Further $J_1 \subseteq U(J_i)$, $2 \leqslant i \leqslant r$, so that we can regard $\chi_2 + \cdots + \chi_r$ as a linear combination of standard characters of $T = S/S^1J_1S^1$. Since $J_2, \ldots, J_r$ are the only regular $\mathscr{J}$-classes of T, induction or repetition shows that $\chi_i = 0$, $2 \leqslant i \leqslant r$. Hence $\chi_i = 0$, $i = 1, \ldots, r$.

THEOREM 5.5. *Every character of a finite semigroup can be uniquely expressed as an integral linear combination of standard irreducible characters.*

Proof. Let S be a finite semigroup and let $J_1, \ldots, J_r$ be as in Lemma 5.4. If χ is a character of S then, by Lemma 5.4, $\chi = \chi_1 + \cdots + \chi_r$ where χ_i is a linear combination of standard irreducible characters with apex J_i, $1 \leqslant i \leqslant r$.

Since J_r is a maximal regular $\mathscr{J}$-class, $J_i \subseteq U(J_r)$, $1 \leqslant i \leqslant r-1$, so that $\chi(x) = \chi_1(x) + \cdots + \chi_{r-1}(x)$ for each $x \in U(J_r)$. Because $J_1, \ldots, J_{r-1}$ are the regular $\mathscr{J}$-classes of $U(J_r)$, $\bar{\chi}_i = \chi_i \mid U(J_r)$ is a linear combination of standard irreducible characters of $U(J_r)$, $1 \leqslant i \leqslant r-1$. Thus, by induction on r, since $\chi \mid U(J_r) = \bar{\chi}_1 + \cdots + \bar{\chi}_r$, the $\bar{\chi}_i$ are integral linear combinations of standard irreducible characters of $U(J_r)$. Hence, since the standard irreducible characters of $U(J_r)$ are linearly independent, it follows that χ_i is an integral linear combination of standard irreducible characters of S, $1 \leqslant i \leqslant r-1$. Then $\chi_r = \chi - (\chi_1 + \cdots + \chi_{r-1}) \in \operatorname{ch} S$ and so $\chi_r \mid H_r \in \operatorname{ch} H_r$, where H_r is a maximal subgroup of J_r. But $\chi_r = \alpha_1\psi_s{}^1 + \cdots + \alpha^t\psi_s{}^t$, where $\psi^1, \ldots, \psi^t$ are the irreducible characters of H_r, then implies $\chi_r \mid H_r = \alpha_1\psi^1 + \cdots + \alpha_t\psi^t$; thus $\alpha_1, \ldots, \alpha_t$ are integers. Hence χ_r is also an integral linear combination of standard irreducible characters and the result holds.

Let G be a subgroup of a finite semigroup S and let R be the $\mathscr{R}$-class of S containing G. Then G acts on R by left multiplication. Let k_α, $\alpha \in A$ be representatives of the orbits of G under this action and, for each $x \in S$, let $M(s)$ be the $A \times A$ matrix over G° with α, β entry

$$M(x)_{\alpha,\beta} = \begin{cases} g & \text{if } k_\alpha x = gk_\beta,\ g \in G \\ 0 & \text{otherwise.} \end{cases}$$

Then a proof like that of the Schützenberger theorem [2, Theorem 3.16] shows that M; $x \to M(x)$ is a representation of S by row-monomial matrices over G°. Any proper representation γ of G thus gives rise to a proper representation Γ of S defined by

$$\Gamma(x) = \gamma(M(x)) \qquad \text{for each } x \in S;$$

we call Γ the *representation of S induced by γ*. If S is a group, this definition of induced representations coincides with the usual one; cf. [3, p. 314]. When G is a maximal subgroup of S, Γ is just the standard extension γ_s of γ to S; cf. [4].

Let H be the maximal subgroup of S containing G; then the induced representation Γ of S has character χ_Γ given by

$$\chi_\Gamma(x) = \sum_\alpha \chi_\gamma(k_\alpha x k_\alpha^{-1}),$$

where χ_γ is extended to H° by defining $\chi_\gamma(g) = 0$ if $g \notin G$; χ_Γ is independent of the particular choice of representatives k_α, $\alpha \in A$ for the orbits of G.

The standard characters of a finite semigroup are the characters of representations of S induced from representations of the maximal subgroups of S. Thus, Theorem 5.5 shows that every character of S is an integral linear combination of characters induced from characters of the maximal subgroups of S. On the other hand, Brauer [1] has shown that every character of a finite group G is an integral linear combination of characters induced from linear characters of elementary subgroups of G. The next proposition shows that we can use Brauer's result to obtain the analog for semigroups.

PROPOSITION 5.6. *Let G be a subgroup of a finite semigroup S and let H be the maximal subgroup of S which contains G. Suppose that χ is a character of G and denote by χ^* and $\bar{\chi}$ the characters of S and H respectively induced by χ. Then $\chi^* = \bar{\chi}_s$.*

Proof. Let q_λ, $\lambda \in \Lambda$ be representatives of the $\mathscr{H}$-classes contained in the $\mathscr{R}$-class of H with $q_1 = e$ the identity of H and let k_α, $\alpha \in A$ be representatives of the cosets of G in H. Then, it is easy to see that $k_\alpha q_\lambda$ are representatives of the orbits of G.

Let $x \in S$; then

$$\bar{\chi}_s(x) = \sum_\Lambda \bar{\chi}(q_\lambda x q_\lambda^{-1}).$$

Now $\bar{\chi}(h) = \sum_\alpha \chi(k_\alpha h k_\alpha^{-1})$ where χ is extended to S by defining $x(y) = 0$ if $y \in S \backslash G$. Hence,

$$\bar{\chi}_s(x) = \sum_{\alpha,\lambda} \chi(k_\alpha q_\lambda x q_\lambda^{-1} k_\alpha^{-1}).$$

On the other hand,

$$\begin{aligned}
\chi^*(x) &= \sum \{\chi(g) : k_\alpha q_\lambda x = g k_\alpha q_\lambda\,, g \in G\} \\
&= \sum \{\chi(g) : q_\lambda x = (k_\alpha^{-1} g k_\alpha)\, q_\lambda\,, g \in G\} \\
&= \sum \{\chi(g) : g = k_\alpha q_\lambda x q_\lambda^{-1} k_\alpha^{-1}\,, g \in G\} \\
&= \sum_{\alpha,\lambda} \chi(k_\alpha q_\lambda x q_\lambda^{-1} k_\alpha^{-1}) = \bar{\chi}_s(x)
\end{aligned}$$

since $\chi(y) = 0$ for $y \notin G$.

THEOREM 5.7. *Every character of a finite semigroup S is an integral linear combination of characters induced from linear characters of the elementary subgroups of S.*

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